

# List-coloring graphs on surfaces with varying list-sizes

Alice M. Dean

Department of Mathematics and Computer Science  
Skidmore College  
Saratoga Springs, NY 12866, USA  
adean@skidmore.edu

Joan P. Hutchinson

Department of Mathematics, Statistics, and Computer Science  
Macalester College  
St. Paul, MN 55105, USA  
hutchinson@macalester.edu

**In Memory of Herbert S. Wilf, 1931-2012**

## Abstract

Let  $G$  be a graph embedded on a surface  $S_\varepsilon$  with Euler genus  $\varepsilon > 0$ , and let  $P \subset V(G)$  be a set of vertices mutually at distance at least 4 apart. Suppose all vertices of  $G$  have  $H(\varepsilon)$ -lists and the vertices of  $P$  are precolored, where  $H(\varepsilon) = \left\lfloor \frac{7+\sqrt{24\varepsilon+1}}{2} \right\rfloor$  is the Heawood number. We show that the coloring of  $P$  extends to a list-coloring of  $G$  and that the distance bound of 4 is best possible. Our result provides an answer to an analogous question of Albertson about extending a precoloring of a set of mutually distant vertices in a planar graph to a 5-list-coloring of the graph and generalizes a result of Albertson and Hutchinson to list-coloring extensions on surfaces.

**Keywords:** list-coloring; Heawood number; graphs on surfaces

# 1 Introduction

For a graph  $G$  the distance between vertices  $x$  and  $y$ , denoted  $\text{dist}(x, y)$ , is the number of edges in a shortest  $x$ - $y$ -path in  $G$ , and we denote by  $\text{dist}(P)$  the least distance between two vertices of  $P$ . In [1] M. O. Albertson asked if there is a distance  $d > 0$  such that every planar graph with a 5-list for each vertex and a set of precolored vertices  $P$  with  $\text{dist}(P) \geq d$  has a list-coloring that is an extension of the precoloring of  $P$ . In that paper he proved such a result for 5-coloring with  $d \geq 4$ , answering a question of C. Thomassen. There have been some preliminary answers to Albertson's question in [4, 8, 11]; initially Tuza and Voigt [17] showed that  $d > 4$ . Kawarabayashi and Mohar [11] have shown that when  $P$  contains  $k$  vertices, there is a function  $d_k > 0$  that suffices for such list-coloring. Then recently Dvořák, Lidický, Mohar and Postle [9] have announced a complete solution, answering Albertson's question in the affirmative, independent of the size of  $P$ .

Let  $S_\varepsilon$  denote a surface of Euler genus  $\varepsilon > 0$ . Its Heawood number is given by

$$H(\varepsilon) = \left\lfloor \frac{7 + \sqrt{24\varepsilon + 1}}{2} \right\rfloor$$

and gives the best possible bound on the chromatic number of  $S_\varepsilon$  except for the Klein bottle whose chromatic number is 6. (For all basic chromatic and topological graph theory results, see [10, 13].) In many instances results for list-coloring graphs on surfaces parallel classic results on surface colorings. Early on it was noted that the Heawood number also gives the list-chromatic number for surfaces; see [10] for history. Also Dirac's Theorem [7] has been generalized to list-coloring by Böhme, Mohar and Stiebitz for most surfaces; the missing case,  $\varepsilon = 3$ , was completed by Král' and Škrekovski. This result informs and eases much of our work.

**Theorem 1.1** ([5, 12]). *If  $G$  embeds on  $S_\varepsilon$ ,  $\varepsilon > 0$ , then  $G$  can be  $(H(\varepsilon) - 1)$ -list-colored unless  $G$  contains  $K_{H(\varepsilon)}$ .*

Analogously to Albertson's question on the plane, we and others (see [11]) ask related list-coloring questions for surfaces. In this paper we ask if there is a distance  $d > 0$  such that every graph on  $S_\varepsilon$ ,  $\varepsilon > 0$ , with  $H(\varepsilon)$ -lists on each vertex and a set of precolored vertices  $P$  with  $\text{dist}(P) \geq d$  has a list-coloring that is an extension of the precoloring of  $P$ . In [3] Albertson and Hutchinson proved the following result; the main result of this paper generalizes this theorem to list-coloring.

**Theorem 1.2** ([3]). *For each  $\varepsilon > 0$ , except possibly for  $\varepsilon = 3$ , if  $G$  embeds on a surface of Euler genus  $\varepsilon$  and if  $P$  is a set of precolored vertices with  $\text{dist}(P) \geq 6$ , then the precoloring extends to an  $H(\varepsilon)$ -coloring of  $G$ .*

Others have studied similar extension questions with  $k$ -lists on vertices for  $k \geq 5$ . For example, see [16], Thm. 4.4, for  $k \geq 6$  and [11], Thm. 6.1, for  $k = 5$ ; however, in both results the embedded graphs must satisfy constraints depending on the Euler genus and the number of precolored vertices. Our main result is Thm. 1.3, which shows that there is a constant bound on the distance between precolored vertices that ensures list-colorability for all graphs embedded on all surfaces when vertices have  $H(\varepsilon)$ -lists. It improves on Thm. 1.2 by removing the possible exception for  $\varepsilon = 3$ , reducing the distance of the precolored vertices from 6 to 4, and broadening the results to list-coloring.

**Theorem 1.3.** *Let  $G$  embed on  $S_\varepsilon$ ,  $\varepsilon > 0$ , and let  $P \subset V(G)$  be a set of vertices with  $\text{dist}(P) \geq 4$ . Then if the vertices of  $P$  each have a 1-list and all other vertices have an  $H(\varepsilon)$ -list,  $G$  can be list-colored. The distance bound of 4 is best possible.*

When  $G$  is embedded on  $S_\varepsilon$ , let the *width* [2] denote the length of a shortest noncontractible cycle of  $G$ ; this is also known as *edge-width*. For list-coloring we have the following corollary of Thms. 1.1 and 1.3.

**Corollary 1.4.** *If  $G$  embeds on  $S_\varepsilon$ ,  $\varepsilon > 0$ , with width at least 4, if the vertices of  $P \subset V(G)$  have 1-lists and all other vertices have  $H(\varepsilon)$ -lists, then  $G$  is list-colorable when  $\text{dist}(P) \geq 3$ . The distance bound of 3 is best possible.*

Given that graphs embedded with very large width can be 5-list-colored as proved in [6], it is straightforward to deduce a 6-list-coloring extension result for such graphs. When  $G$  embeds on  $S_\varepsilon$ ,  $\varepsilon > 0$ , with width at least  $2^{O(\varepsilon)}$ , if a set of vertices  $P$  with  $\text{dist}(P) \geq 3$  have 1-lists and all others have 6-lists, then after the vertices of  $P$  are deleted and the color of each  $x \in P$  is deleted from the lists of  $x$ 's neighbors, the remaining graph has 5-lists, large width, and so is list-colorable. Thus  $G$  is list-colorable, but only when embedded with large width whose size increases with the Euler genus of the surface.

A consequence of Thomassen's proof of 5-list-colorability of planar graphs [15] is that if all vertices of a graph in the plane have 5-lists except that the vertices of one face have 3-lists, then the graph can be list-colored. For surfaces, we offer as a related result another corollary of Thm. 1.3.

**Corollary 1.5.** *If  $G$  embeds on  $S_\varepsilon$ ,  $\varepsilon > 0$ , and contains a set of faces each pair of which is at distance at least two apart, with all vertices on these faces having  $(H(\varepsilon) - 1)$ -lists and all other vertices having  $H(\varepsilon)$ -lists, then  $G$  can be list-colored.*

The paper concludes with related questions.

## 2 Background results on surfaces, Euler genus and the Heawood formula

Let  $S_\varepsilon$  denote a surface of Euler genus  $\varepsilon > 0$ . If  $\varepsilon$  is odd, then  $S_\varepsilon$  is the nonorientable surface with  $\varepsilon$  crosscaps, but when  $\varepsilon$  is even,  $S_\varepsilon$  may be orientable or not. We let  $\mathcal{T}$  denote the torus, the orientable surface of Euler genus 2, and  $\mathcal{K}$  the Klein bottle, the nonorientable surface of Euler genus 2.

The Heawood number  $H(\varepsilon)$ , defined above, gives the largest  $n$  for which  $K_n$  embeds on a surface  $S_\varepsilon$  of Euler genus  $\varepsilon$ , as well as the chromatic number of  $S_\varepsilon$ , except that  $K_6$  is the largest complete graph embedding on  $\mathcal{K}$  and 6 is its chromatic number.

The least Euler genus  $\varepsilon$  for which  $K_n$  embeds on  $S_\varepsilon$  is given by the inverse function

$$\varepsilon = I(n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil.$$

Each  $K_n$ ,  $n \geq 5$ , of course, has a minimum value of  $\varepsilon > 0$  for which it embeds on  $S_\varepsilon$ , called the *Euler genus of  $K_n$* , but for  $\varepsilon \geq 2$  more than one surface  $S_\varepsilon$  may have the same maximum  $K_n$  that embeds on it. For example, both  $S_5$  and  $S_6$  have Heawood number 9 with  $K_9$  being the largest complete graph that embeds there. Embedding patterns of  $K_{H(\varepsilon)}$  depend on the congruence class of  $H(\varepsilon)$  modulo 3 for  $\varepsilon \geq 1$ . In Table 1, which gives values of  $\varepsilon$  and  $H(\varepsilon)$  for  $\varepsilon = 1, \dots, 24$ ,  $e$  is the number of edges in  $K_{H(\varepsilon)}$ ,  $f = 2 - \varepsilon - v + e$  is the number of faces in a 2-cell embedding of  $K_{H(\varepsilon)}$  on  $S_\varepsilon$ , and the final column gives the size of the largest possible face when  $K_{H(\varepsilon)}$  is so embedded. That largest face size is three more than the difference  $2e - 3f$ .

For our results we need to know when  $K_{H(\varepsilon)}$  necessarily has a 2-cell embedding on  $S_\varepsilon$ . When  $K_n$  embeds on  $S_\varepsilon$ , but not on  $S_{\varepsilon-1}$ , then  $K_n$  necessarily embeds with a 2-cell embedding. When  $K_n$  embeds in addition on  $S_{\varepsilon+1}, \dots, S_{\varepsilon+i}$  with  $i > 0$ , then it may not have a 2-cell embedding on the latter surfaces. For example, on surfaces  $S_1$ ,  $\mathcal{T}$ ,  $S_4$ , and  $S_5$ , the complete graphs

$\varepsilon$	$H(\varepsilon)$	$e$	$f$	Largest Face	$\varepsilon$	$H(\varepsilon)$	$e$	$f$	Largest Face
1	6	15	10	3	13	12	66	43	6
2	7	21	14	3	14	12	66	42	9
3	7	21	13	6	15	13	78	52	3
4	8	28	18	5	16	13	78	51	6
5	9	36	24	3	17	13	78	50	9
6	9	36	23	6	18	13	78	49	12
7	10	45	30	3	19	14	91	60	5
8	10	45	29	6	20	14	91	59	8
9	10	45	28	9	21	14	91	58	11
10	11	55	36	5	22	15	105	70	3
11	11	55	35	8	23	15	105	69	6
12	12	66	44	3	24	15	105	68	9

Table 1: Embedding parameters for  $K_{H(\varepsilon)}$

$K_6, K_7, K_8$  and  $K_9$  have 2-cell embeddings, respectively, but  $K_6, K_7$  and  $K_9$  may or may not have 2-cell embeddings on  $\mathcal{K}$ ,  $S_3$  and  $S_6$ , respectively.

If  $f$  is a face of an embedded graph  $G$ , let  $V(f)$  and  $E(f)$  denote the incident vertices and edges of  $f$ . We say that  $V(f) \cup E(f)$  is the *boundary* of  $f$  and that the *closure* of  $f$  is the union of  $f$  and its boundary. Each edge of  $E(f)$  either lies on another face besides  $f$  or it might lie just on  $f$ . For example, Fig. 1 shows two graphs embedded on the torus,  $\mathcal{T}$ . In the first graph, edges 2-3 and 4-7 each border two faces, but edges 3-6 and 8-9 each border only one face. The size  $s$  of a face  $f$  is determined by counting, with multiplicity, the number of edges on its boundary, and we then call  $f$  an  $s$ -region. In other words, when  $s_1$  edges of  $E(f)$  lie on another face of  $G$  besides  $f$  and  $s_2$  edges lie only on  $f$ , then we call  $f$  an  $s$ -region where  $s = s_1 + 2s_2$ . When  $f$  is a 2-cell,  $E(f)$  forms a single facial walk  $W_f$ , and the size of the face equals the length of the facial walk, counting multiplicity of repeated edges. Since an  $s$ -region  $f$  may have repeated edges and repeated vertices, we indicate  $|V(f)| = t$  by calling  $f$  also a  $t$ -vertex-region where  $t \leq s$ . Hence the shaded region in the first graph in Fig. 1 is a 13-region and a 9-vertex-region, since two edges and four vertices are repeated; the shaded region in the second graph, with no repeated vertices or edges, is a 13-region

and a 13-vertex-region.

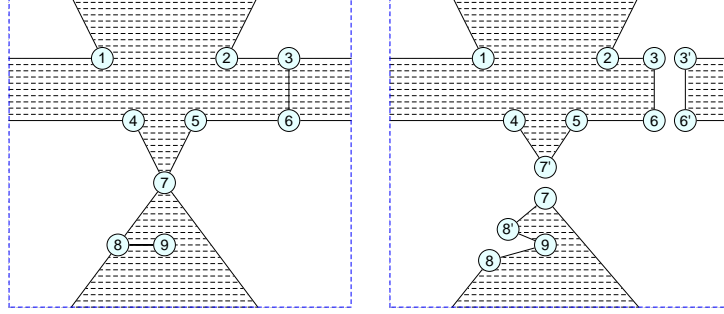


Figure 1: A 2-cell region in a graph embedded on the torus,  $\mathcal{T}$ , before and after vertex- and edge-duplication

Here in summary are statistics on 2-cell embeddings of  $K_{H(\varepsilon)}$ . The patterns presented are visible from Table 1 and are easily derived from Euler's formula and the function  $I(n)$ , given above.

**Lemma 2.1.** *Let  $\varepsilon \geq 1$  and suppose  $K_{H(\varepsilon)}$  has a 2-cell embedding on  $S_\varepsilon$  (but  $S_\varepsilon \neq \mathcal{K}$ ). Set  $i = \left\lfloor \frac{H(\varepsilon)-3}{3} \right\rfloor$  so that  $H(\varepsilon) = 3i + 3, 3i + 4$  or  $3i + 5$  with  $i \geq 1$ .*

1. *If  $H(\varepsilon) = 3i + 3$ , then  $\varepsilon = (3i^2 - i)/2, (3i^2 - i + 2)/2, \dots$ , or  $(3i^2 + i - 2)/2$ . The number of faces of the embedding is given by  $f = 3i^2 + 5i + 2, 3i^2 + 5i + 1, \dots$ , or  $3i^2 + 4i + 3$ , respectively, and the largest possible face is an  $s$ -region with  $s = 3, 6, \dots$ , or  $3i$ , resp.*
2. *If  $H(\varepsilon) = 3i + 4$ , then  $\varepsilon = (3i^2 + i)/2, (3i^2 + i + 2)/2, \dots$ , or  $(3i^2 + 3i)/2$ . The number of faces of the embedding is given by  $f = 3i^2 + 7i + 4, 3i^2 + 7i + 3, \dots$ , or  $3i^2 + 6i + 4$ , respectively, and the largest possible face is an  $s$ -region with  $s = 3, 6, \dots$ , or  $3i + 3$ , resp.*
3. *If  $H(\varepsilon) = 3i + 5$ , then  $\varepsilon = (3i^2 + 3i + 2)/2, (3i^2 + 3i + 4)/2, \dots$ , or  $(3i^2 + 5i)/2$ . The number of faces of the embedding is given by  $f = 3i^2 + 9i + 6, 3i^2 + 9i + 5, \dots$ , or  $3i^2 + 8i + 7$ , respectively, and the largest possible face is an  $s$ -region with  $s = 5, 8, \dots$ , or  $3i + 2$ , resp.*

From the point of view of the genus, given  $\varepsilon > 0$ , we can determine directly whether or not  $K_{H(\varepsilon)}$  necessarily has a 2-cell embedding on  $S_\varepsilon$ .  $K_{H(\varepsilon)}$  necessarily has a 2-cell embedding if and only if  $\varepsilon = (3i^2 - i)/2$  or  $(3i^2 + i)/2$

or  $(3i^2 + 3i + 2)/2$  for some value of  $i > 0$ . Thus given  $\varepsilon > 0$ , we compute  $H(\varepsilon)$  and set  $i = \lfloor H(\varepsilon)/3 \rfloor - 1$  so that  $H(\varepsilon) = 3i + 3, 3i + 4$ , or  $3i + 5$ . Then  $K_{H(\varepsilon)}$  necessarily embeds with a 2-cell embedding if  $I(H(\varepsilon)) = \varepsilon$ ; that is,  $S_\varepsilon$  is the genus surface for  $K_{H(\varepsilon)}$ .

In the results of Table 1 we do not claim that every 2-cell embedding of  $K_{H(\varepsilon)}$  achieves the maximum face size when that size is greater than three. For example when  $K_{H(\varepsilon)}$  has a largest face being a 5- or 6-region, it might embed as a near-triangulation with one 5- or 6-region, respectively, or it might be a triangulation except for two 4-regions or a triangulation except for a 4- and a 5-region, resp. (An embedding is a *near-triangulation* if at most one region is not 3-sided.)

We note from Table 1 and Lemma 2.1 that there are some instances of  $\varepsilon$  when  $K_{H(\varepsilon)}$  embeds possibly with an  $(H(\varepsilon) - 1)$ -region which might allow for the embedding of two different (not disjoint, but distinct) copies of  $K_{H(\varepsilon)}$  on  $S_\varepsilon$ , as explained in the next lemma.

**Lemma 2.2.** *Let  $K_{H(\varepsilon)}$  have a 2-cell embedding on  $S_\varepsilon$ ,  $\varepsilon > 0$ .*

1. *The largest possible face in the embedding is an  $(H(\varepsilon) - 1)$ -region. If there is an  $(H(\varepsilon) - 1)$ -region, there is just one, and the embedding is a near-triangulation.*
2. *If every face of the embedding is at most an  $(H(\varepsilon) - 2)$ -region, then no additional copy of  $K_{H(\varepsilon)}$  can simultaneously embed on  $S_\varepsilon$ .*
3. *When  $K_{H(\varepsilon)}$  can embed with an  $(H(\varepsilon) - 1)$ -region that is also an  $(H(\varepsilon) - 1)$ -vertex-region, then two different copies of  $K_{H(\varepsilon)}$  can embed, by adding a vertex adjacent to all vertices of that region, and then the two complete graphs share a copy of  $K_{H(\varepsilon)-1}$ . Such an embedding is possible only if  $H(\varepsilon) = 3i + 4$  and  $\varepsilon = (3i^2 + 3i)/2$ , and the resulting embedding is a triangulation.*

We call the latter graph  $DK_{H(\varepsilon)}$ ; it is also  $K_{H(\varepsilon)+1} \setminus \{e\}$  for some edge  $e$ .

*Proof.* Suppose that  $K_{H(\varepsilon)}$  has a 2-cell embedding with at least one  $s$ -region where  $s \geq H(\varepsilon) - 1$ . Then Euler's formula plus a count of edges on faces with multiplicities leads to a contradiction to Lemma 2.1 in all cases except when there is precisely one  $(H(\varepsilon) - 1)$ -region,  $H(\varepsilon) = 3i + 4$ ,  $\varepsilon = (3i^2 + 3i)/2$ , and all other faces are 3-regions.

Suppose  $K_{H(\varepsilon)}$  embeds on  $S_\varepsilon$  with every face having at most  $H(\varepsilon) - 2$  sides. No two additional vertices in different faces of  $K_{H(\varepsilon)}$  can be adjacent. For  $2 \leq k \leq 4$ ,  $k$  mutually adjacent, additional vertices cannot form  $K_{H(\varepsilon)}$  together with  $H(\varepsilon) - k$  vertices on the boundary of a face.

Proofs of remaining parts follow easily from Euler's Formula and Lemma 2.1.  $\square$

If  $V' \subseteq V(G)$ , we denote by  $G[V']$  the induced subgraph on the vertices in  $V'$ ; for  $E' \subseteq E(G)$ , we denote by  $G[E']$  the induced subgraph on the edge set  $E'$ . When  $f$  is a face of an embedded  $G$ , we may also call the subgraph  $G[E(f)]$  the boundary of  $f$ ; that is, it may be convenient at times to think of the boundary of a face  $f$  as a set  $V(f) \cup E(f)$  and at other times as the subgraph  $G[E(f)]$ .

We restate two very useful corollaries of Thm. 6 in [5]. The first involves a case that is not covered in that theorem, but which follows easily from their proof. If  $f$  is the infinite face of a connected plane graph, we call the boundary of  $f$  the *outer boundary* of  $G$ , and when  $G[E(f)]$  is a cycle, we call it the *outer cycle*. Without loss of generality we may suppose that for a connected plane graph the outer boundary is a cycle.

**Corollary 2.3.** ([5]) *Let  $G$  be a connected plane graph with outer cycle  $C$  that is a  $k$ -cycle with  $k \leq 6$ . If every vertex of  $G$  has a list of size at least 6, then a precoloring of  $C$  extends to all of  $G$  unless  $k = 6$ , there is a vertex in  $V(G) \setminus V(C)$  that is adjacent to all vertices of  $V(C)$ , and its list consists of six colors that appear on the precolored  $C$ .*

Then the results of Thm. 6 in [5] together with Cor. 2.3 give the next corollary.

**Corollary 2.4** ([5]). *Let  $G$  be a connected plane graph with outer cycle  $C$  that is a  $k$ -cycle with  $3 \leq k \leq 6$ . If every vertex of  $G$  has a list of size at least  $\max(5, k + 1)$ , then a precoloring of  $C$  extends to all of  $G$ .*

The next lemma is used repeatedly in the proof of Thms. 3.3 and 4.3. It is an extension of the similar result for 5-list-colorings in [5]. The parameters are motivated by the ‘‘Largest Face’’ and  $H(\varepsilon)$ -list sizes from Table 1.

**Lemma 2.5.** *Let  $H$  be a connected graph with a 2-cell embedding on  $S_\varepsilon$ ,  $\varepsilon > 0$ , and let  $f$  be a 2-cell  $k$ -region of  $H$ ,  $k \geq 3$ . Let  $G$  be a plane graph embedded within  $f$  and let  $G_f$  be a simple, connected graph that consists*



of  $G$ ,  $H[E(f)]$ , and edges joining  $V(G)$  and  $V(f)$  so that  $G_f$  is embedded in the closure of  $f$ . Let  $P = \{v_1, \dots, v_j\}$  be a subset of  $V(G_f)$  satisfying  $\text{dist}(P) \geq 3$ . Then if every vertex of  $G_f$  has an  $\ell$ -list except that the vertices of  $P$  each have a 1-list, every proper precoloring of  $H[E(f)]$  extends to a list-coloring of  $G_f$  provided that no vertex of  $P$  is adjacent to a vertex of  $V(f)$  with the same color as its 1-list, and

1.  $k = 3$  and  $\ell \geq 6$ ,
2.  $k \geq 4$  and  $\ell \geq k + 2$ , or
3.  $k = 6$  or  $k \geq 9$ ,  $\ell = k + 1$ , and there is no vertex  $x$  adjacent to  $k + 1$  vertices of  $V(f) \cup \{v_i\}$ , for some  $i = 1, \dots, j$ , with  $x$ 's list consisting of  $\ell = k + 1$  colors that all appear on  $V(f) \cup \{v_i\}$ .

*Proof.* Note that  $G_f[E(f)] = H[E(f)]$ . Also note that the condition  $\text{dist}(P) \geq 3$  guarantees that no vertex of  $G_f$  is adjacent to more than one  $v_i$ . For  $v_i \in P \setminus V(f)$ , we say that we *excise*  $v_i$  if we delete it and delete its color from the list of colors for each neighbor that is not precolored. The proof has three cases that together prove parts 1-3 of the lemma.

Case A. Assume  $k = 3$  and  $\ell \geq 6$ ,  $4 \leq k \leq 6$  and  $\ell \geq k + 2$ , or  $k = 6$  and  $\ell = 7$ . In these cases first we excise the vertices of  $P \setminus V(f)$  so that every remaining vertex of  $G$  has a list of size at least 5 for  $k = 3$ , of size at least  $k + 1$  for  $k = 4, 5, 6$ , or else of size at least 6 when  $k = 6$ .

In the following we may need to do some surgery, perhaps repeatedly, on the face  $f$  and its boundary, so that we can apply Cor. 2.4. First, more easily, when  $f$  is a 2-cell  $k$ -region on which lies no repeated vertex, then  $G_f$  is a plane graph with outer cycle a  $k$ -cycle,  $k \leq 6$ . By Cor. 2.4 a precoloring of  $G_f[E(f)]$  extends to  $G_f \setminus P$  and this coloring extends to all of  $G_f$  unless there is a vertex  $x$  with a 6-list, adjacent to six vertices of  $V(f)$  with the six colors of  $x$ 's list. If  $x$ 's list was decreased to a 6-list,  $x$  was adjacent to some vertex  $v_i$ , but this situation is disallowed by hypothesis in part 3.

Otherwise in a traversal of  $W_f$  we visit a vertex more than once and may travel along an edge twice. In the former case, each time we revisit a vertex  $x$ , we can split that vertex in two, into  $x_1$  and  $x_2$ , and similarly divide the edges incident with  $x$  so that the face  $f$  is expanded to become the new face  $f'$ , still a  $k$ -region, and the graph  $G_f$  becomes  $G_{f'}$  which is naturally embedded in the closure of  $f'$  and contains the same adjacencies. Now there is one more vertex in  $V(f')$  and the same set of edges  $E(f') = E(f)$  on

the boundary and in the boundary subgraph  $G_{f'}[E(f')]$ . A precoloring of  $G_f[E(f)]$  gives a precoloring of  $G_{f'}[E(f')]$  in which vertices  $x_1$  and  $x_2$  receive the same color; we call this procedure *vertex-duplication*. In the latter case, when we revisit an edge  $e = (y, y')$ , we may visit both of its endpoints twice or one endpoint twice and the other just once. We similarly duplicate the edge  $e = (y, y')$  by duplicating one or both of its endpoints and splitting  $e$  into two new edges  $e_1$  and  $e_2$ . Then we divide the other edges incident with  $e$  so that  $G_f$  becomes  $G_{f'}$  which is naturally embedded in the closure of the new face  $f'$ , still a  $k$ -region, but now with one or two more vertices in  $V(f')$ , the same number of edges in  $E(f')$  and in  $G_{f'}[E(f')]$ , and with one less duplicated edge in  $W_{f'}$ . A precoloring of  $G_f[E(f)]$  gives a precoloring of  $G_{f'}[E(f')]$  in which duplicated vertices receive the same color; we call this procedure *edge-duplication*. We note that in both duplications there cannot be a vertex  $x$  that is adjacent to both copies of a duplicated vertex (since  $G_f$  is a simple graph). As an example, the first graph in Fig. 1 shows a 2-cell face that is a 13-region, in which vertices 3, 6, 7, and 8, are repeated, and edges 3-6 and 8-9 are repeated. Vertex- and edge-duplication produces the second graph, which has a new face that is a 13-region and whose facial walk is a cycle given by 1-8-9-8'-7-2-3-6-5-7'-4-6'-3'-1.

In all cases after vertex- and edge-duplication, the 2-cell  $k$ -region  $f$  becomes a 2-cell  $k$ -region  $f^*$  with no repeated vertex or edge on the outer boundary.  $G_f$  has been transformed into a plane graph  $G_{f^*}$  with outer cycle,  $G_{f^*}[E(f^*)]$ , of length  $k \leq 6$ . The precoloring of  $G_f[E(f)]$  has become a precoloring of  $G_{f^*}[E(f^*)]$  with duplicated vertices receiving the same color. Then by Cor. 2.4, the precoloring of  $G_{f^*}[E(f^*)]$  extends to  $G_{f^*} \setminus P$  and so the precoloring of  $G_f[E(f)]$  extends to  $G_f \setminus P$  and to all of  $G_f$  since the exceptional case of part 3 cannot occur. (Since  $f^*$  is at most a 6-region and has a duplicated vertex, it is a  $t$ -vertex-region for some  $t < 6$ , and there cannot be a vertex adjacent to six vertices of  $V(f^*)$ .)

Case B. Suppose  $k \geq 7$  and  $\ell \geq k + 2$  so that in all cases  $\ell \geq 9$ . For  $v \in V(G)$ , let  $E_f(v)$  denote the set of edges joining  $v$  with a vertex of  $V(f)$ . Suppose there is a vertex  $x$  of  $V(G)$  that is adjacent to at least  $k - 3$  vertices of  $V(f)$ . If  $x = v_i$  for some  $i, 1 \leq i \leq j$ , then  $G_f[E(f) \cup E_f(v_i)]$  can be properly colored by assumption. If  $x \neq v_i$  for any  $i, 1 \leq i \leq j$ , then  $x$  is adjacent to either one or no vertex  $v_i$ , and since  $x$  has an  $\ell$ -list,  $\ell \geq k + 2$ , the coloring of  $G_f[E(f) \cup E_f(v_i)]$  (respectively,  $G_f[E(f)]$ ) extends to  $x$ . In all cases  $G_f[E(f) \cup E_f(x)]$  divides  $f$  into regions of size at most 6, and the coloring of  $G_f[E(f) \cup E_f(x)]$  extends to the interior of each  $s$ -region,

$3 \leq s \leq 6$ , by Case A since interior vertices, other than the  $v_i$ , have 9-lists.

Otherwise every vertex  $x$  in  $G$  is adjacent to at most  $k - 4$  vertices of  $V(f)$ . For each such vertex  $x$  we delete from its list the colors of  $V(f)$  to which it is adjacent. This may reduce the list for  $x$  to one of size six or more. Next we excise the vertices of  $P$  in  $G \setminus V(f)$ , resulting in the planar graph  $G \setminus P$  with every vertex having a list of size at least five, which can be list-colored by [15]. This list-coloring is compatible with the precoloring of  $G_f[E(f)]$  and extends to  $P$  and so to all of  $G_f$ .

Case C. The case of  $k = 6, \ell = 7$  was covered in Case A. Suppose that  $k \geq 9$  and  $\ell = k + 1 \geq 10$ . Suppose there is a vertex  $x$  of  $V(G)$  that is adjacent to at least  $k - 4$  vertices of  $V(f)$ . As before, if  $x = v_i$  for some  $i, 1 \leq i \leq j$ , then  $G_f[E(f) \cup E_f(v_i)]$  can be properly colored by assumption. If  $x \neq v_i$  for any  $i, 1 \leq i \leq j$ , then  $x$  is adjacent to one or no vertex  $v_i$ , and the coloring of  $G_f[E(f) \cup E_f(v_i)]$  (resp.,  $G_f[E(f)]$ ) extends to  $x$  in all cases unless (since  $\ell = k + 1$ )  $x$  is adjacent to all vertices of  $V(f) \cup \{v_i\}$  for some  $i, 1 \leq i \leq j$ , and  $x$ 's list consists of  $\ell$  colors all appearing on  $V(f) \cup \{v_i\}$ . We have disallowed this case. Now  $G_f[E(f) \cup E_f(x)]$  forms a graph that consists of triangles and  $s$ -regions with  $s \leq 7$ . The coloring of  $G_f[E(f) \cup E_f(x)]$  extends to the interior of each region by the previous cases, since  $\ell \geq 10$ .

Otherwise every vertex  $x$  of  $G$  is adjacent to at most  $k - 5$  vertices of  $V(f)$ , and we proceed as in the proof of Case B by decreasing the lists of vertices adjacent to  $V(f)$  and excising all the  $v_i$  to create a planar graph with every vertex having at least a 5-list. The resulting graph is list-colorable with a coloring compatible with that of  $G_f[E(f)]$  and extending to  $G_f$ .  $\square$

### 3 Results on $K_n$ genus surfaces

Most parts of the proof of the next lemma are clear; these results are used repeatedly in the proof of the main results.

- Lemma 3.1.** *1. Suppose at most one vertex of  $K_n$  has a 1-list, at least one vertex has an  $n$ -list, and the remaining vertices have  $(n - 1)$ -lists or  $n$ -lists. Then  $K_n$  can be list-colored.*
- 2. If one vertex of  $DK_n$  has a 1-list and all other vertices have  $n$ -lists, then  $DK_n$  can be list-colored.*
- 3. If at most six vertices of  $DK_n$ ,  $n \geq 7$ , have lists of size  $n - 1$  and all others have  $n$ -lists, then  $DK_n$  can be list-colored.*

*Proof.* We include the proof of part 3. Suppose that one of the two vertices of degree  $n - 1$ , say  $x$ , has an  $n$ -list. Then  $K_n = DK_n \setminus \{x\}$  has at most six vertices with  $(n - 1)$ -lists and can be list-colored since  $n \geq 7$ . This coloring extends to  $x$  which has an  $n$ -list and is adjacent to  $n - 1$  vertices of the colored  $K_n$ . Otherwise both vertices of degree  $n - 1$ , say  $x$  and  $y$ , have  $(n - 1)$ -lists,  $L(x)$  and  $L(y)$  respectively. Suppose there is a common color  $c$  in  $L(x)$  and  $L(y)$ . Then coloring  $x$  with  $c$  extends to a coloring of  $K_n = DK_n \setminus \{y\}$  after which  $y$  can also be colored with  $c$ . Otherwise  $L(x)$  and  $L(y)$  are disjoint. Suppose that when  $DK_n \setminus \{y\}$  is list-colored, the colors on  $K_{n-1} = DK_n \setminus \{x, y\}$  are precisely the  $n - 1$  colors of  $L(y)$  so that the coloring does not extend. If there is some vertex  $z$  of  $K_{n-1}$  with an  $n$ -list that contains a color not in  $L(y)$  and different from the color  $c_x$  used on  $x$ , we use  $c_x$  on  $z$ , freeing up the previous color of  $z$  for  $y$ . Otherwise, for every  $z$  with an  $n$ -list, that list equals  $L(y) \cup \{c_x\}$ . Besides these vertices of  $K_{n-1}$  with prescribed  $n$ -lists, there are at most four others in  $K_{n-1}$  which have  $n - 1$  lists. These four vertices might be colored with colors from  $L(x)$ , but that still leaves at least one color  $c'_x \neq c_x$  in  $L(x)$  that has not been used. We change the color of  $x$  to  $c'_x$  and the color of one of the  $n$ -list vertices of  $K_{n-1}$  to  $c_x$ , thus freeing up that vertex's previous color to be used on  $y$ .  $\square$

**Theorem 3.2.** *Suppose  $G$  embeds on  $S_\varepsilon$ ,  $\varepsilon > 0$ , and does not contain  $K_{H(\varepsilon)}$ . Then when every vertex of  $G$  has an  $H(\varepsilon)$ -list except that the  $j$  vertices of  $P = \{v_1, \dots, v_j\}$ ,  $j \geq 0$ , have 1-lists and  $\text{dist}(P) \geq 3$ , then  $G$  is list-colorable.*

*Proof.* Let  $G$  embed on  $S_\varepsilon$ ,  $\varepsilon > 0$ , and suppose  $G$  does not contain  $K_{H(\varepsilon)}$ . We excise the vertices of  $P = \{v_1, \dots, v_j\}$ , if present, leaving a graph with all vertices having at least  $(H(\varepsilon) - 1)$ -lists since  $\text{dist}(P) \geq 3$ . By [5, 12], the smaller graph can be list-colored, and that list-coloring extends to  $G$ .  $\square$

In particular this result holds for all graphs on the Klein bottle since  $K_7$  does not embed there. The first value not covered by the next theorem is  $\varepsilon = 3$  with  $H(\varepsilon) = 7$ .

**Theorem 3.3.** *Suppose  $G$  has a 2-cell embedding on  $S_\varepsilon$ ,  $\varepsilon > 0$ , and contains  $K_{H(\varepsilon)}$ . Then when every vertex of  $G$  has an  $H(\varepsilon)$ -list except that the  $j$  vertices of  $P = \{v_1, \dots, v_j\}$ ,  $j \geq 0$ , have 1-lists,  $G$  is list-colorable provided that  $\varepsilon$  is of the form  $\varepsilon = (3i^2 - i)/2$ ,  $(3i^2 + i)/2$ , or  $(3i^2 + 3i + 2)/2$ , for some  $i \geq 1$ , and  $\text{dist}(P) \geq 4$ .*

*Proof.* We know that  $K_{H(\varepsilon)}$  necessarily has a 2-cell embedding on  $S_\varepsilon$  for  $\varepsilon = 1, 4$  as does  $K_7$  on  $\mathcal{T}$ . ( $K_6$  and  $K_7$  may or may not have 2-cell embeddings on  $\mathcal{K}$  and on  $S_3$ , respectively.)

The values  $\varepsilon = (3i^2 - i)/2$ ,  $(3i^2 + i)/2$ , or  $(3i^2 + 3i + 2)/2$  for some  $i \geq 1$  are those for which  $K_{H(\varepsilon)}$  necessarily has a 2-cell embedding on  $S_\varepsilon$ ; they give the value of the genus surface of  $K_{H(\varepsilon)}$  for each of the modulo 3 classes of  $H(\varepsilon)$ . Since  $\text{dist}(P) \geq 4$ , at most one vertex  $v_k \in P$  is in or is adjacent to a vertex of  $K_{H(\varepsilon)}$  (but not both), and in the latter case  $v_k$  is adjacent to at most  $H(\varepsilon) - 1$  vertices of the complete graph since  $K_{H(\varepsilon)+1}$  does not embed on  $S_\varepsilon$ . Thus in all cases  $K_{H(\varepsilon)} \cup P$  can be list-colored by Lemma 3.1.1. When  $\varepsilon = 1$ ,  $H(\varepsilon) = 6$ , and  $K_6$  embeds as a triangulation on  $S_1$ . When  $\varepsilon > 1$ , if  $\varepsilon = (3i^2 - i)/2$  or  $(3i^2 + i)/2$ ,  $K_{H(\varepsilon)}$  embeds as a triangulation, and if  $\varepsilon = (3i^2 + 3i + 2)/2$ ,  $K_{H(\varepsilon)}$  embeds with the largest face size at most five, and in all cases  $H(\varepsilon) \geq 7$ . Hence we apply Lemma 2.5 for  $\varepsilon \geq 1$  to see that the list-coloring of  $K_{H(\varepsilon)}$  extends to the interior of each of its faces and so  $G$  is list-colorable.  $\square$

A similar proof would show that when the orientable surface  $S_\varepsilon$  with  $\varepsilon$  even is the orientable genus surface for  $K_{H(\varepsilon)}$  (i.e., when  $\varepsilon$  is even and gives the least Euler genus such that  $K_{H(\varepsilon)}$  embeds on orientable  $S_\varepsilon$ ), then for every  $G$  with a 2-cell embedding on orientable  $S_\varepsilon$  and containing  $K_{H(\varepsilon)}$  the same list-coloring result holds. The first corollary of Section 1 also follows easily.

*Proof of Cor. 1.4.* Suppose  $H(\varepsilon) = 3i + 3, i \geq 1$ . If  $\varepsilon = (3i^2 - i)/2$ , then  $K_{H(\varepsilon)}$  embeds with  $f = (i + 1)(3i + 2)$  faces by Lemma 2.1.1.  $K_{H(\varepsilon)}$  contains  $(3i + 3)(3i + 2)(3i + 1)/6$  3-cycles, more than the number of faces so that  $K_{H(\varepsilon)}$  embeds with a noncontractible 3-cycle. Thus in this case  $G$  cannot contain  $K_{H(\varepsilon)}$  and by Thm. 3.2,  $G$  can be list-colored. If  $\varepsilon = (3i^2 - i + 2)/2, \dots$ , or  $(3i^2 + i - 2)/2$ , then  $K_{H(\varepsilon)}$  embeds with fewer than  $f = (i + 1)(3i + 2)$  faces and so the same result holds.

When  $H(\varepsilon) = 3i + 4$  or  $3i + 5, i \geq 1$ , an analogous proof shows that  $G$  cannot contain  $K_{H(\varepsilon)}$  and so is list-colorable.

To see that distance at least 3 is best possible for the precolored vertices, take a vertex  $x$  with a  $k$ -list  $L(x)$  and attach  $k$  pendant edges to vertices, precolored with each of the colors of  $L(x)$ .  $\square$

## 4 All surfaces

First we explore some topology of surfaces and non-2-cell faces of embedded graphs. Cycles on surfaces (i.e., simple closed curves on the surface), for both orientable and nonorientable surfaces, are of three types: contractible and surface-separating, noncontractible and surface-separating, and noncontractible and surface-nonseparating. (When the meaning is clear, we suppress the prefix “surface.”) A non-2-cell face of an embedded graph must contain a noncontractible surface cycle within its interior. For example, in the second graph in Fig. 1, the shaded region is a 2-cell face, and the unshaded region is a non-2-cell face that contains a noncontractible and nonseparating cycle. (For a more detailed discussion see Chapters 3 and 4 of [13].)

Suppose  $f$  is a non-2-cell face of  $K_{H(\varepsilon)}$  embedded on  $S_\varepsilon$ . We repeatedly “cut” along simple noncontractible surface cycles that lie wholly within the face  $f$  until the “derived” face or faces become 2-cells. Each “cut” is replaced with one or two disks, creating a new surface, and with each “cut”  $K_{H(\varepsilon)}$  stays embedded on a surface  $S_{\varepsilon'}$  with  $\varepsilon' < \varepsilon$ . Below we explain this surface surgery and count the number of newly created faces, called *derived* faces in the surgery.

**Lemma 4.1.** *Suppose  $K_{H(\varepsilon)}$  embeds on  $S_\varepsilon$ ,  $\varepsilon > 0$ . Then the largest possible 2-cell face in the embedding is an  $(H(\varepsilon) - 1)$ -region.*

*Proof.* Suppose the embedded  $K_{H(\varepsilon)}$  has a non-2-cell  $k$ -region  $f$ ; initially there are no derived faces. In  $f$  we can find a simple noncontractible cycle  $C$ , disjoint from its boundary,  $V(f) \cup E(f)$ . If  $C$  is surface-separating, it is necessarily 2-sided. We replace  $C$  by two copies of itself,  $C$  and  $C'$ , and insert in each copy a disk, producing surfaces  $S(1)$  and  $S'(1)$ , each with Euler genus that is positive and less than  $\varepsilon$ . Since  $K_{H(\varepsilon)}$  is connected, it is embedded on one of these surfaces, say  $S(1)$ . The face  $f$  of  $K_{H(\varepsilon)}$  on  $S_\varepsilon$  becomes the derived face  $f_1$  of  $K_{H(\varepsilon)}$  on  $S(1)$  and retains the same set of boundary vertices  $V(f_1) = V(f)$  and edges  $E(f_1) = E(f)$  so that  $f_1$  is also a  $k$ -region. Initially  $f$  is not a derived face,  $f_1$  becomes a derived face and the Euler genus decreases by at least 1. If, later on in the process,  $f$  is a derived face, then  $f_1$  is also a derived face, the number of derived faces does not increase, and the Euler genus decreases by at least 1.

If  $C$  is not surface-separating and is 2-sided, we duplicate it and sew in two disks, as above, to create one new surface  $S(1)$  of lower and positive Euler genus on which  $K_{H(\varepsilon)}$  is embedded. If  $C$  was not separating within the

face  $f$ , then the derived face  $f_1$  keeps the same set of boundary vertices and edges as  $f$  and remains a  $k$ -region. As above, the number of derived faces increases by at most 1 and the Euler genus decreases by at least 2. If  $C$  was separating within the face  $f$ , then  $f$  splits into two derived faces  $f_1$  and  $f'_1$ . Each vertex of  $V(f)$  and each edge of  $E(f)$  appears on one of these derived faces or possibly two when it was a repeat on  $f$ . More precisely, if  $f_1$  is a  $k_1$ -region and  $f'_1$  is a  $k'_1$ -region, then necessarily  $k_1 + k'_1 = k$ . In this case the Euler genus decreases by 2 and number of derived faces increases by at most 2, increasing by 2 only when the face being cut was an original face of  $K_{H(\varepsilon)}$ . If  $C$  is not surface-separating and is 1-sided, we replace  $C$  by a cycle  $DC$  of twice the length of  $C$  and insert a disk within  $DC$ , producing a surface  $S(1)$  with Euler genus that is less than  $\varepsilon$ .  $K_{H(\varepsilon)}$  remains embedded on  $S(1)$ , necessarily with positive Euler genus, and the derived face  $f_1$  keeps the same boundary vertices and edges as  $f$ , remaining a  $k$ -region. Thus the number of derived faces increases by at most 1 and the Euler genus decreases by at least 1.

Now we prove the lemma by induction on the number of non-2-cell faces of the embedded  $K_{H(\varepsilon)}$ . We know the conclusion holds when there are no non-2-cell faces by Lemma 2.2. Otherwise let  $f$  be a non-2-cell  $k$ -region. We repeatedly cut along simple noncontractible cycles within  $f$  and its derived faces, creating surfaces  $S(1), S(2), \dots$  on which  $K_{H(\varepsilon)}$  remains embedded. We continue until every derived face of  $f$  is a 2-cell. Then  $K_{H(\varepsilon)}$  is embedded on, say,  $S_{\varepsilon'}$  with  $\varepsilon' < \varepsilon$  and has fewer non-2-cell faces. By induction each 2-cell face has size at most  $H(\varepsilon) - 1$  and thus every original 2-cell face, which has not been affected by the surgery, also has size at most  $H(\varepsilon) - 1$ .  $\square$

We have purposefully proved more within the previous proof.

**Corollary 4.2.** *Suppose  $K_{H(\varepsilon)}$  has a non-2-cell embedding on  $S_\varepsilon$ , and suppose that after cutting along noncontractible cycles in non-2-cell faces,  $K_{H(\varepsilon)}$  has a 2-cell embedding on  $S_{\varepsilon'}$ ,  $\varepsilon' < \varepsilon$ . Then the number of faces in the latter embedding that are derived from faces in the original embedding is at most  $\varepsilon - \varepsilon'$ .*

*Proof.* In the previous proof we saw that with some cuts the number of derived faces is increased by at most 1 and the Euler genus is decreased by at least 1; let  $c_0$  denote the number of cuts in which there is no increase in the number of derived faces and  $c_1$  the number of cuts in which there is an increase of 1 in the number of derived faces. If the increase is always at

most 1, then the result follows. The number of derived faces is increased by 2 precisely when the cutting cycle  $C$  within a face  $f'$  is 2-sided, is not surface-separating, is separating within  $f'$ , and  $f'$  is an original face of the embedding. In that case the Euler genus is decreased by 2 also; let  $c_2$  denote the number of such cuts. Then the decrease in the Euler genus,  $\varepsilon - \varepsilon'$  is at least  $c_0 + c_1 + 2c_2 \geq c_1 + 2c_2$ , which equals the number of derived faces.  $\square$

**Theorem 4.3.** *Given  $\varepsilon > 0$  and  $G$  a graph on  $n$  vertices that has a 2-cell embedding on  $S_\varepsilon$ , suppose that  $G$  contains  $K_{H(\varepsilon)}$ . If  $P \subset V(G)$  satisfies  $\text{dist}(P) \geq 4$ , then if the vertices of  $P$  each have a 1-list and every other vertex of  $G$  has an  $H(\varepsilon)$ -list, then  $G$  can be list-colored.*

*Proof.* The proof is by induction on  $\varepsilon$  and on  $n$ . We know the theorem holds for  $G$  with a 2-cell embedding on  $S_\varepsilon$  for  $1 \leq \varepsilon \leq 2$  by Thm. 3.3. Consider graphs with 2-cell embeddings on  $S_{\varepsilon^*}$  for  $\varepsilon^* \geq 3$ . For each such embedded graph, the subgraph  $K_{H(\varepsilon^*)}$  inherits an embedding on  $S_{\varepsilon^*}$ , and  $H(\varepsilon^*) \geq 7$ .

Since  $\text{dist}(P) \geq 4$  we know that at most one vertex of  $P$  lies in or is adjacent to a vertex of  $K_{H(\varepsilon^*)}$ . If there is one, call it  $v_i^*$  and if not, ignore reference to  $v_i^*$  in the following. By Lemma 3.1.1 we know that  $G[V(K_{H(\varepsilon^*)}) \cup \{v_i^*\}]$  can be list-colored since  $v_i^*$  is adjacent to at most  $H(\varepsilon^*) - 1$  vertices of  $K_{H(\varepsilon^*)}$  (because  $K_{H(\varepsilon^*)+1}$  does not embed on  $S_{\varepsilon^*}$ ). If  $G$  contains a vertex  $x$  in neither  $V(K_{H(\varepsilon^*)})$  nor  $P$ , then  $G[V(K_{H(\varepsilon^*)}) \cup \{x\}]$  can be list-colored by first coloring  $K_{H(\varepsilon^*)}$  and then coloring  $x$ , which has an  $H(\varepsilon^*)$ -list and is adjacent to at most  $H(\varepsilon^*) - 1$  vertices of  $K_{H(\varepsilon^*)}$ .

Thus on surface  $S_{\varepsilon^*}$  we know the result holds for every graph on  $n$  vertices with  $n \leq H(\varepsilon^*) + 1$ . Let  $G$  have  $n^*$  vertices,  $n^* > H(\varepsilon^*) + 1$ , and have a 2-cell embedding on  $S_{\varepsilon^*}$ .

Let  $f$  be a  $k$ -region in the inherited embedding of  $K_{H(\varepsilon^*)}$  with incident vertices  $V(f)$  and edges  $E(f)$ , and let  $G_f$  denote the subgraph of  $G$  lying in the closure of  $f$ ,  $f \cup V(f) \cup E(f)$ . Suppose  $f$  is a 2-cell face of  $K_{H(\varepsilon^*)}$  in whose interior lie vertices of  $V(G) \setminus \{V(f) \cup \{v_i^*\}\}$ ; call these interior vertices  $U_f$ . Then after deleting the vertices of  $U_f$ ,  $G \setminus U_f$  has a 2-cell embedding on  $S_{\varepsilon^*}$  with fewer than  $n^*$  vertices, contains  $K_{H(\varepsilon^*)}$ , and contains vertices of  $P' \subseteq P$  with  $\text{dist}(P') \geq 4$ . By induction  $G \setminus U_f$  is list-colorable. By Lemma 4.1  $k \leq H(\varepsilon^*) - 1$ . We claim that the resulting list-coloring of  $G[V(f) \cup \{v_i^*\}]$  extends to  $G_f$ .

If  $k \leq H(\varepsilon^*) - 2$ , then the coloring extends by Lemma 2.5.1 and 2.5.2. Otherwise  $k = H(\varepsilon^*) - 1$  and the coloring then extends by Lemma 2.5.3, unless there is a vertex  $x$  of  $G_f$  that has an  $H(\varepsilon^*)$ -list, is adjacent to  $v_i^*$ , not



in  $V(f)$ , and to all vertices of  $V(f)$ , and its  $H(\varepsilon^*)$ -list consists of  $H(\varepsilon^*)$  colors that appear on its neighbors. Then  $G[V(K_{H(\varepsilon^*)}) \cup \{x\}]$  forms  $DK_{H(\varepsilon^*)}$ , which triangulates  $S_{\varepsilon^*}$  and does not contain another vertex of  $P$  since  $\text{dist}(P) \geq 4$ . Since  $v_i^*$  is adjacent to at most three vertices of  $DK_{H(\varepsilon^*)}$  (the vertices of a 3-region),  $G[V(DK_{H(\varepsilon^*)}) \cup \{v_i^*\}]$  can be list-colored by Lemma 3.1.3. Then the list-coloring extends to the graph in the interior of each 3-region by Lemma 2.5.1 since  $H(\varepsilon^*) \geq 7$ .

Thus we can assume that every vertex of  $V(G) \setminus \{V(K_{H(\varepsilon^*)}) \cup \{v_i^*\}\}$  lies in a non-2-cell region of the embedding of  $K_{H(\varepsilon^*)}$  on  $S_{\varepsilon^*}$ . We claim there are two vertices of  $K_{H(\varepsilon^*)}$  that lie only on its 2-cell faces; we prove that below. One of these might lie in  $P$  or be adjacent to  $v_i^*$ , but the other, say  $x^*$ , has an  $H(\varepsilon^*)$ -list and is adjacent only to vertices of  $K_{H(\varepsilon^*)}$ , precisely  $H(\varepsilon^*) - 1$  of these.

In that case we consider  $G \setminus \{x^*\}$ . If  $G \setminus \{x^*\}$  does not contain  $K_{H(\varepsilon^*)}$ , it can be list-colored by Thm. 3.2. Otherwise  $G \setminus \{x^*\}$  does contain  $K_{H(\varepsilon^*)}$ .  $G \setminus \{x^*\}$  might have a 2-cell embedding on  $S_{\varepsilon^*}$  or it might not. In the former case, by induction on  $n$  it can be list-colored. Suppose that  $G \setminus \{x^*\}$  does not have a 2-cell embedding on  $S_{\varepsilon^*}$ . Then the face  $f^*$  that was formed by deleting  $x^*$  is the one and only non-2-cell face of that embedding since no other face of  $G$  has been changed by the deletion of  $x^*$ . Then we cut along noncontractible cycles within  $f^*$ , as described in Lemma 4.1, until every face, derived from  $f^*$ , is a 2-cell in  $G \setminus \{x^*\}$  now embedded on  $S_{\varepsilon'}$  with  $\varepsilon' < \varepsilon^*$ . We have  $H(\varepsilon') = H(\varepsilon^*)$  since  $G \setminus \{x^*\}$  contains  $K_{H(\varepsilon^*)}$ . Thus  $G \setminus \{x^*\}$  can be list-colored by induction on the Euler genus, and in all cases that coloring extends to  $G$  since  $x^*$  has a list of size  $H(\varepsilon^*)$  which is larger than its degree.

We return to the claim that there are two vertices of  $K_{H(\varepsilon^*)}$  that lie only on 2-cell faces of its embedding on  $S_{\varepsilon^*}$ , given that every vertex of  $V(G) \setminus \{V(K_{H(\varepsilon^*)}) \cup \{v_i^*\}\}$  lies in a non-2-cell face of the embedded  $K_{H(\varepsilon^*)}$ . Since the number of vertices of  $G$ ,  $n^*$ , is greater than  $H(\varepsilon^*) + 1$ , there are some non-2-cell faces containing other vertices of  $G$ . We count the maximum number of vertices of  $K_{H(\varepsilon^*)}$  that lie on these non-2-cells to show that number is at most  $H(\varepsilon^*) - 2$ .

As in Lemma 4.1 we repeatedly cut each non-2-cell face of the embedded  $K_{H(\varepsilon^*)}$  until all remaining faces, the original and the derived, are 2-cells; suppose  $K_{H(\varepsilon^*)}$  is then embedded on  $S_{\varepsilon'}$  with  $\varepsilon' < \varepsilon^*$ . We know that every vertex originally on a non-2-cell face of  $K_{H(\varepsilon^*)}$  is represented on at least one derived face and we show below that the total number of vertices on derived faces is at most  $H(\varepsilon^*) - 2$ . We also know that  $\varepsilon' \geq I(H(\varepsilon^*))$ . Let

$n_1 = \varepsilon' - I(H(\varepsilon^*))$ , which is nonnegative, and  $n_2 = \varepsilon^* - \varepsilon'$ , which is positive. The variable  $n_1$  will determine the face sizes in the 2-cell embedding of  $K_{H(\varepsilon^*)}$  on  $S_{\varepsilon'}$  (see Table 1), and  $n_2$  will determine the maximum number of derived faces that have been created.

We consider the modulo 3 class of  $H(\varepsilon^*)$ , and we begin with the case of  $H(\varepsilon^*) = 3i + 4$ ,  $i \geq 1$ . We know that  $\varepsilon^* \in \{(3i^2 + i)/2, \dots, (3i^2 + 3i)/2\} = \{I(3i + 4), \dots, I(3i + 4) + i\}$  so that  $n_1 + n_2 \leq i$  by Lemma 2.1. By Cor. 4.2 the number of derived faces is at most  $n_2$ . We can determine the possible face sizes of a 2-cell embedding of  $K_{H(\varepsilon^*)}$  on  $S_{\varepsilon'}$  with  $\varepsilon' = I(3i + 4) + n_1$ . A 2-cell embedding on  $S_{I(3i+4)}$  is necessarily a triangulation. A 2-cell embedding on  $S_{I(3i+4)+1}$  consists of triangles except possibly for one 6-region, or triangles plus two faces whose sizes sum to 9, or triangles plus three faces whose sizes sum to 12 (necessarily three 4-regions). More generally when  $\varepsilon' = I(3i + 4) + n_1$ , then the embedding might consist of triangles plus one  $(3n_1 + 3)$ -region, or triangles plus two faces whose sizes sum to  $3n_1 + 6$ , or triangles plus three faces whose sizes sum to  $3n_1 + 9$ , etc. And if we choose  $n_2$  faces, all the derived faces, the sum of their sizes can be at most  $3n_1 + 3n_2 \leq 3i < 3i + 2 = H(\varepsilon^*) - 2$ .

For  $i \geq 1$ , the same calculation holds when  $H(\varepsilon^*) = 3i + 3$ , and when  $H(\varepsilon^*) = 3i + 5$ , a similar count will work. In the latter case we have  $n_1 + n_2 \leq i - 1$ , though the face sizes may be slightly larger. A 2-cell embedding of  $K_{H(\varepsilon^*)}$  on  $S_{I(3i+5)}$  may have triangles plus a 5-region or triangles plus two 4-regions. In general a 2-cell embedding of  $K_{H(\varepsilon^*)}$  on  $S_{I(3i+5)+n_1}$  might have triangles plus one  $(3n_1 + 5)$ -region or triangles plus two regions whose sizes sum to  $3n_1 + 8$ , etc. With  $n_2$  faces, all the derived faces, their sum of sizes can be at most  $3n_1 + 3n_2 + 2 \leq 3i - 1 < 3i + 3 = H(\varepsilon^*) - 2$ .  $\square$

We now complete the proof our main result, Thm. 1.3.

*Proof of Thm. 1.3.* If  $G$  has a non-2-cell embedding on  $S_{\varepsilon}$  that contains  $K_{H(\varepsilon)}$ , we can perform surgery on the non-2-cell faces, as we did in the proof of Lemma 4.1 and Thm. 4.3, to obtain a 2-cell embedding of  $G$  on a surface of Euler genus  $\varepsilon' < \varepsilon$  that still contains  $K_{H(\varepsilon)}$ , and hence  $H(\varepsilon') = H(\varepsilon)$ . We can thus apply Thm. 4.3 to  $G$  on  $S_{\varepsilon'}$ . This shows that the result holds for every embedding, 2-cell or non-2-cell, and Thm. 1.3 follows.  $\square$

The distance bound of 4 in Thms. 1.3 and 4.3 is best possible, for consider  $K_{H(\varepsilon)}$  with a pendant edge attaching a degree-1 vertex to each vertex of  $K_{H(\varepsilon)}$ . Give each degree-1 vertex the list  $\{1\}$  and place that vertex in the set

$P$ . When every other vertex has an identical  $H(\varepsilon)$ -list that contains 1, the graph is not list-colorable and  $\text{dist}(P) = 3$ .

The second corollary of Section 1 now follows easily.

*Proof of Cor. 1.5.* Let  $f_1, \dots, f_j$  be the faces with vertices with smaller lists. Add a vertex  $x_i$  to  $f_i$  and make it adjacent to all vertices of  $V(f_i)$ . Give each  $x_i$  a 1-list  $\{\alpha\}$  where  $\alpha$  appears in no list of a vertex of  $G$ , and add  $\alpha$  to the list of each vertex of  $V(f_i)$ , now the neighbors of  $x_i$ . Then  $G \cup \{x_1, \dots, x_j\}$  can be list-colored by Thm. 1.3 since with  $P = \{x_1, \dots, x_j\}$ ,  $\text{dist}(P) \geq 4$ , and this coloring is a list-coloring of  $G$ .  $\square$

## 5 Concluding Questions

1. Škrekovski [14] has shown the extension of Dirac's theorem that if  $G$  is embedded on  $S_\varepsilon$ ,  $\varepsilon \geq 5$ ,  $\varepsilon \neq 6, 9$ , and does not contain  $K_{H(\varepsilon)-1}$  or  $K_{H(\varepsilon)-4} + C_5$ , then  $G$  can be  $(H(\varepsilon) - 2)$ -colored. Is the same true for list-coloring?
2. If  $G$  embeds on  $S_\varepsilon$  and does not contain one of the two graphs of Question 1, if the vertices of one face have at least  $(H(\varepsilon) - 2)$ -lists, and if all other vertices have at least  $H(\varepsilon)$ -lists, can  $G$  be list-colored?

## Acknowledgements

We wish to thank D. Archdeacon and a referee for helpful comments and Z. Dvořák and K.-I. Kawarabayashi for information on background material.

## References

- [1] M. O. Albertson. You can't paint yourself into a corner. *J. Combin. Theory Ser. B*, 73:189–194, 1998.
- [2] M. O. Albertson and J. P. Hutchinson. The independence ratio and genus of a graph. *Trans. Amer. Math. Soc.*, 226:161–173, 1977.
- [3] M. O. Albertson and J. P. Hutchinson. Graph color extensions: when Hadwiger's conjecture and embeddings help. *Electron. J. Combinatorics*, 9:R37, 2002.

- [4] M. Axenovich, J. P. Hutchinson, and M. A. Lastrina. List precoloring extension in planar graphs. *Discrete Math.*, 311:1046–1056, 2011.
- [5] T. Böhme, B. Mohar, and M. Stiebitz. Dirac’s map-color theorem for choosability. *J. Graph Theory*, 32:327–339, 1998.
- [6] M. DeVos, K.-I. Kawarabayashi, and B. Mohar. Locally planar graphs are 5-choosable. *J. Combin. Theory Ser. B*, 98:1215–1232, 2008.
- [7] G. Dirac. Short proof of a map-colour theorem. *Canad. J. Math.*, 9:225–226, 1957.
- [8] Z. Dvořák, B. Lidický, and B. Mohar. 5-choosability of graphs with crossings far apart. Manuscript, 2011.
- [9] Z. Dvořák, B. Lidický, B. Mohar, and L. Postle. 5-list-coloring planar graphs with distant precolored vertices. Manuscript, 2011.
- [10] T. Jensen and B. Toft. *Graph Coloring Problems*. John Wiley and Sons, New York, USA, 1995.
- [11] K.-I. Kawarabayashi and B. Mohar. List-color-critical graphs on a fixed surface. In *Proceedings of the twentieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1156–1165, 2009. SODA ’09.
- [12] D. Král’ and R. Škrekovski. The last excluded case of Dirac’s map-color theorem for choosability. *J. Graph Theory*, 51:319–354, 2006.
- [13] B. Mohar and C. Thomassen. *Graphs on Surfaces*. Johns Hopkins University Press, Baltimore, MD, USA, 2001.
- [14] R. Škrekovski. A theorem on map colorings. *Bull. of Inst. of Comb. and Applic.*, 35:53–60, 2002.
- [15] C. Thomassen. Every planar graph is 5-choosable. *J. Combin. Theory Ser. B*, 62(1):180–181, 1994.
- [16] C. Thomassen. Color-critical graphs on a fixed surface. *J. Combin. Theory Ser. B*, 70:67–100, 1997.
- [17] Z. Tuza and M. Voigt. A note on planar 5-list-colouring: non-extendability at distance 4. *Discrete Math.*, 251:169–172, 2002.